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Unit 3, 79 High Street, Pershore, Worcestershire. WR10 1EX. Tel: (044) 05603121643
www.polycomm.uk.com

Pythagorean Triples and Fermat's Last Theorem.

Author: Greg Spring. C. Eng B.Sc MIEE. (Institute of Electrical Engineers)
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1.0) Introduction.

In 1985, the BBC's flag ship science programme, Horizon, broadcast a programme about unsolved theories and problems which included Fermat's Last Theorem of 1637 in which Fermat* stated that no integer solutions existed for the equation:

$a^n = b^n + c^n$,   (Eq 1); if n was an integer whose value was greater than 2.

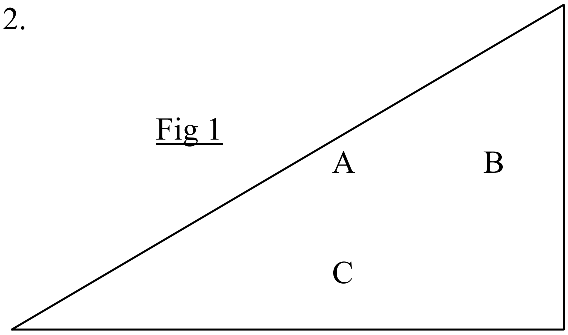
Intrigued by the simplicity of the equation and the fact that, at the time, no one had found or rediscovered Fermat's alleged general proof, the author made various attempts to solve Fermat's Last Theorem. Although no claim is made to have either rediscovered Fermat's proof nor offer an alternative, analysis of Fermat's original problem has allowed a novel technique to be formulated which allows the automatic generation of Pythagorean triples from a seed integer. Examples include $5^2 = 4^2 + 3^2$ and $13^2 = 12^2 + 5^2$.

*[Pierre de Fermat. Born Montauban, France, 1601. Died Castres, France, 12th January 1665.]

2.0) History.

It should be noted that this abstract equation is a multi dimensional equation depending upon the value of 'n' and refers to a right angled triangle when n = 2, as illustrated below for clarity. In this example sides A, B & C are mathematically related by the equation $A^2 = B^2 + C^2$, i.e. $a^n = b^n + c^n$ when n = 2.

Fig 1



The reader will quickly appreciate that when n = 1, ($a^n = b^n + c^n$) is reduced to just (a = b + c), an expression which has an infinite number of real number and integer solutions. The reader will remember from school maths lessons that when n = 2, ($a^n = b^n + c^n$) becomes the famous Pythagorean equation for right angled triangles, ($a^2 = b^2 + c^2$) from which can be calculated the hypotenuse of a right angled triangle providing two lengths or one length and an angle are known.

Famous Pythagorean triples include $5^2 = 4^2 + 3^2$ and $13^2 = 12^2 + 5^2$, triples which were known to the ancient Sumerians and Babylonians four thousand years ago because they used the formula ($p^2 - q^2$), 2pq, ($p^2 + q^2$) to calculate perfect triples. The reader can quickly determine this formula works by substituting p = 2, q = 1 or p = 3, q = 2 for example, but it is all the more remarkable that the Babylonians also worked out that 2291, 2700 and 3541 is also a perfect triplet using this method.

It is not known why the Sumerians and Babylonians needed to know these perfect triples, other than for, perhaps, agricultural mapping, but regardless of the exact reason, several perfect integer triples have been known to man for at least four thousand years and knowledge of their existence has always fascinated mankind for practical and aesthetic reasons.

Pierre de Fermat is the first person recorded in history to seek a general solution for the abstract equation $a^n = b^n + c^n$ and whilst solutions for n = 1 and n = 2 were known for a long time, it would have seemed odd to Pierre Fermat and other mathematicians throughout history that there appeared to be no known integer solutions for the cubic volume equation, $a^3 = b^3 + c^3$ or the fourth order space equation, $a^4 = b^4 + c^4$ or even higher order triples. Pierre Fermat was curious to understand this problem and went on to prove that there were no integer solutions for the two specific equations when n = 3 and n = 4. Whilst the proof for n = 3 is lost, n = 4 is extant and provided an insight into Fermat's analytical methods. His proof was based upon the assumption that three integer solutions existed to satisfy the equation $a^4 = b^4 + c^4$. The analysis went on to show that three smaller integers could also exist which also satisfied the equation and so on, down to the smallest set of a, b and c which clearly did not satisfy the equation. Once a minimum set of a, b and c were found that could not satisfy the equation, then none of the previous values for a, b and c could satisfy the equation because all sets of a, b and c were mathematically linked. This mathematical strategy aimed to prove a flaw or logical inconsistency with the original argument and by proving the flaw, bring down the whole argument like a house of cards. In this way, Fermat proved there were no integer solutions for $a^4 = b^4 + c^4$. Although lost, there is little doubt Fermat would have used the same argument to also prove there are no integer solutions for when n = 3. One of the reasons why Fermat's proof for n = 3 is lost is because Fermat didn't keep very good records of his mathematical research, preferring to scribble notes in the margins of other people's work which he read for enjoyment. Whilst working on a general solution for the equation ($a^n = b^n + c^n$) and translating and editing one of the five extant copies of thirteen books of Arithmetica, written by the Greek mathematician, Diophantus, Fermat is said to have written in a margin: "I have found a truly marvellous demonstration which this margin is too narrow to contain". The demonstration Fermat referred to was a proof that there were no integer solutions for the abstract equation, ($a^n = b^n + c^n$) for all integer values of n > 2. Unfortunately, Fermat either forgot to record his thoughts on this topic or they were lost after his death. Either way, the World was only left with his tantalising note suggesting that he had solved this problem with a seemingly simple and elegant proof.

Whether Fermat did or did not prove that there are no integer solutions for ($a^n = b^n + c^n$) for all values of n > 2 remains a mystery due to the loss of much of his work. History granted Fermat the benefit of doubt for this discovery because he had already proven many theorems and solved the problem for the two discrete cases when n = 3 and n = 4. However, the tantalising note Fermat left behind in 1637, frustrated the attempts of very many well known mathematicians to rediscover Fermat's alleged proof, all without success. French mathematician Legendre proved there were no integer solution for n = 5 in 1823 and in 1840, Lamé and Lebesgue gave proofs for when n = 7. Dirichlet gave a proof for n = 14 and in 1849, Kummer proved that there were no integer solutions for some Bernoullian conditions, 37, 59 and 67 being exceptions. Because Fermat's last theorem defied solution for nearly 360 years, it acquired an extraordinary celebrity status among mathematicians until Princeton mathematician, Professor Andrew Wiles indirectly, but finally proved the theorem in 1995 as a consequence of proving the Taniyama Shimura conjecture.

The reference for this information was obtained from "A short account of the history of mathematics" by W. W. Rouse Ball, Trinity College, Cambridge, an unabridged internet copy of the author's fourth edition published in 1908. International Standard Book Number: 0-486-20630-0. Library of Congress Catalog Card Number: 60-3187

3.0) A new analytical approach.

Although Andrew Wiles proved Fermat's Last Theorem, the following analysis is offered as a bi-product of the authors unsuccessful attempts to solve Fermat's Last Theorem.

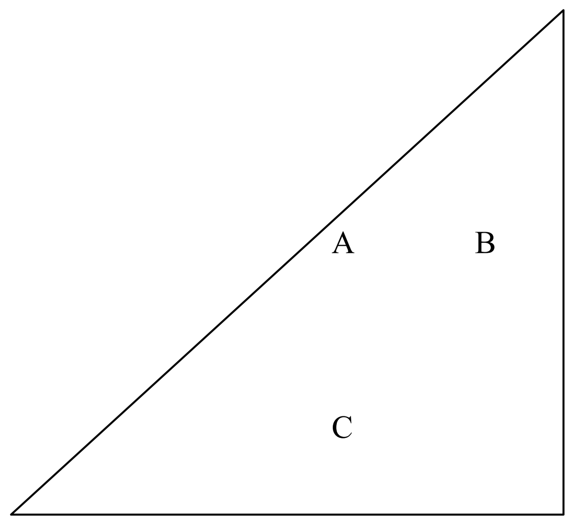
Given the right angled triangle below, we know from Pythagoras that the sides A, B and C are related by the equation; ($A^2 = B^2 + C^2$) and through the centuries many perfect Pythagorean triples have been found, ($5^2 = 4^2 + 3^2$) and ($13^2 = 12^2 + 5^2$) are just two of the most recognisable triples.

Table 1 illustrates a sample of eleven integer triples, although there are infinitely many more.

A	B	C
5	4	3
13	12	5
17	15	8
25	24	7
29	21	20
37	35	12
41	40	9
61	60	11
65	63	16
85	84	13
113	112	15

Table 1

Fig 2



Ancient mathematicians looked for patterns and symmetry in numbers to find an underlying truth or beauty. Using this philosophical approach, we begin this analysis of Pythagorean triples by assuming there is an underlying relationship between the numbers that make up all Pythagorean triples. Since the hypotenuse of all right angles triangles is always the largest number, we start by converting the Pythagorean equation:

($A^2 = B^2 + C^2$) into an equivalent that links the three numbers together, i.e. $A^2 = (A\text{-}p)^2 + (A\text{-}q)^2$

For clarity, we also choose to replace A with n, such that the equation becomes:

$$n^2 = (n - p)^2 + (n - q)^2 \text{ ....(Eqn 2)}$$

Expanding terms, we get: $n^2 = n^2 - 2np + p^2 + n^2 - 2nq + q^2$

Collecting terms, we get: $n^2 = 2n^2 - 2n(p + q) + (p^2 + q^2)$

And hence from this analysis we form a new quadratic equation: $n^2 - 2n(p + q) + (p^2 + q^2) = 0$

Using the standard quadratic formula to solve this equation: $(-b \pm \sqrt{(b^2 - 4ac)})/2a$

The solutions are $n_1, n_2 = (p + q) \pm \sqrt{2pq}$ …………(Eq 3)

The difficulty with solving this solution for p and q is the surd, $\pm \sqrt{2pq}$ which restricts the number of possible integer solutions. To find this limit, we need to ensure $\pm \sqrt{2pq}$ becomes a perfect square, i.e. the values of p and q always ensure 2pq is a perfect square and therefore always produces an integer square root, e.g. 4 roots to 2, 9 roots to 3, 16 roots to 4 and so on.

Let $r = \pm \sqrt{2pq}$, then $r^2 = 2pq$ and making p the subject: $p = r^2 / 2q$

From which it is seen that for all values of q, the denominator, 2q is always even.

This means 'r' cannot be an odd number because $r^2 / 2q$ will always leave a fractional remainder.

This can be shown as follows:

Let 'r' be any generalised odd number, i.e., $r = (2u + 1)$ where 'u' is any integer.

Therefore: $r^2 = (2u + 1)(2u + 1)$ and hence $p = (2u + 1)(2u +1)/ 2q$, i.e. $p = (4u^2 + 4u + 1)/2q$

Any value of 'u' will always leave a remainder equal to 1/2q, which is fractional because q is defined to be an integer. Therefore for a perfect square to exist, 'r' cannot be an odd number. Since all numbers must be either odd or even, 'r' must therefore be an even integer, i.e.

$r = 2u$ ► $p = (2u)^2 / 2q$ ► $p = 4u^2 / 2q$ ► $2pq = 4u^2$ ► $pq = 2u^2$

and we have achieved the goal of expressing 2pq in terms of a perfect square, since $4u^2$ will generate perfect integer roots for all values of 'u'. Table 2 is a small tabulation of the first ten positive integers to illustrate the effect.

u	$4u^2$	$\pm \sqrt{2pq} = \pm\sqrt{4u^2}$
0	0	0
1	4	2
2	16	4
3	36	6
4	64	8
5	100	10
6	144	12
7	196	14
8	256	16
9	324	18
10	400	20

Table 2

Note that 'u' represents any positive, zero or negative seed integer and therefore p and q represent integer factors which can be used to solve the quadratic solution;
$n_1, n_2 = (p + q) \pm \sqrt{2pq}$ and because steps have been taken to ensure 2pq is a perfect square, n_1 and n_2 will therefore always exist as integers.

Factors p and q are actually derived from $2u^2$ because there is a redundant 2 in the $2pq = 4u^2$ relationship. Factors p and q are therefore calculated from $pq = 2u^2$

4.0) Application.

Because the seed integer 'u' is squared, 'u' can be any positive, zero or a negative integer, but for simplicity let u = 1 to begin.

Let u = 1

$2u^2 = 2(1)^2 = 2$. Therefore, non repeating factors of $2u^2$ are $p = 1$, $q = 2$

Note that once the pq factors repeat, the process of factorising the integer, $2u^2$ stops because this would produce redundant factors and therefore redundant information. Using the quadratic solution from page 5, the two possible values of n are:

$$n_1, n_2 = (p + q) \pm \sqrt{2pq} \text{ ....(Eq 3)}$$

$$\text{i.e. } n_1, n_2 = (1 + 2) \pm \sqrt{2(1)2}$$

$$n_1, n_2 = 3 \pm 2 \qquad \text{i.e.} \qquad n_1, n_2 = 5 \text{ or } 1$$

Substituting these integers into Eqn 2, we get:

$$n_1^2 = (n_1 - p)^2 + (n_1 - q)^2 \qquad \text{or} \qquad n_2^2 = (n_2 - p)^2 + (n_2 - q)^2$$

$$5^2 = (5 - 1)^2 + (5 - 2)^2 \qquad \text{or} \qquad 1^2 = (1 - 1)^2 + (1 - 2)^2$$

$$5^2 = 4^2 + 3^2 \qquad \text{or} \qquad 1^2 = 0^2 + (-1)^2$$

And we discover that when the seed integer is unity, two Pythagorean triples are created, but only one is of any real interest. The other is known as a trivial solution because it equates a number to itself and contains zero in the solution.

Let u = 2

$2u^2 = 2(2)^2 = 8$ and the non repeating factors of $2u^2$ are:

p =	1	2
q =	8	4

Now we have four factors when u = 2 which will generate four Pythagorean triples.

Substituting these factors into Eqn 3, we get:

$$n_1, n_2 = (1 + 8) \pm \sqrt{2(1)8} \qquad \text{and} \qquad n_3, n_4 = (2 + 4) \pm \sqrt{2(2)4}$$

$$n_1, n_2 = 9 \pm 4 \qquad \text{and} \qquad n_3, n_4 = 6 \pm 4$$

$$n_1, n_2 = 13 \text{ or } 5 \qquad \text{and} \qquad n_3, n_4 = 10 \text{ or } 2$$

Substituting these integers into Eqn 2, we get:

$n_1^2 = (n_1 - p_1)^2 + (n_1 - q_1)^2$ & $n_2^2 = (n_2 - p_1)^2 + (n_2 - q_1)^2$

& $n_3^2 = (n_3 - p_2)^2 + (n_3 - q_2)^2$ & $n_4^2 = (n_4 - p_2)^2 + (n_4 - q_2)^2$

$13^2 = (13 - 1)^2 + (13 - 8)^2$ & $5^2 = (5 - 1)^2 + (5 - 8)^2$

& $10^2 = (10 - 2)^2 + (10 - 4)^2$ & $2^2 = (2 - 2)^2 + (2 - 4)^2$

$13^2 = 12^2 + 5^2$ & $5^2 = 4^2 + (-3)^2$ & $10^2 = 8^2 + 6^2$ & $2^2 = 0^2 + (-2)^2$

And we discover that when the seed integer, u = 2, it generates three real and one trivial Pythagorean triples, although strictly speaking only one new triple has been calculated because the ($5^2 = 4^2 + 3^2$) triple has been repeated twice in the form of ($5^2 = 4^2 + (-3)^2$) and ($10^2 = 8^2 + 6^2$).

<u>Let u = 3</u>

$2u^2 = 2(3)^2 = 18$ and the non repeating factors of $2u^2$ are:

p =	1	2	3
q =	18	9	6

Now we have six separate possible solutions when u = 3, i.e. 2 x the non repeating factors of pq.

Substituting these integers into Eqn 3, we get:

$n_1, n_2 = (1 + 18) \pm \sqrt{2(1)18}$ & $n_3, n_4 = (2 + 9) \pm \sqrt{2(2)9}$ & $n_5, n_6 = (3 + 6) \pm \sqrt{2(3)6}$

$n_1, n_2 = 19 \pm 6$ & $n_3, n_4 = 11 \pm 6$ & $n_5, n_6 = 9 \pm 6$

$n_1, n_2 = 25$ or 13 & $n_3, n_4 = 17$ or 5 & $n_5, n_6 = 15$ or 3

Substituting these integers into Eq 2, we get:

$n_1, n_2 = n^2 = (n - p_1)^2 + (n - q_1)^2$ & $n_3, n_4 = n^2 = (n - p_2)^2 + (n - q_2)^2$ & $n_5, n_6 = n^2 = (n - p_3)^2 + (n - q_3)^2$

$25^2 = (25 - 1)^2 + (25 - 18)^2$ & $17^2 = (17 - 2)^2 + (17 - 9)^2$ & $15^2 = (15 - 3)^2 + (15 - 6)^2$

$25^2 = 24^2 + 7^2$ & $17^2 = 15^2 + 8^2$ & $15^2 = 12^2 + 9^2$

and

$13^2 = (13 - 1)^2 + (13 - 18)^2$ & $5^2 = (5 - 2)^2 + (5 - 9)^2$ & $3^2 = (3 - 3)^2 + (3 - 6)^2$

$13^2 = 12^2 + (-5)^2$ & $5^2 = 3^2 + (-4)^2$ & $3^2 = 0^2 + (-3)^2$

Of these six results, only two are new triples, ($25^2 = 24^2 + 7^2$) and ($17^2 = 15^2 + 8^2$).
The $13^2 = 12^2 + (-5)^2$ triple is a version of the $13^2 = 12^2 + 5^2$ triple found when u =2.
The ($15^2 = 12^2 + 9^2$) triple is just three times the $5^2 = 4^2 + 3^2$ triple found when u =1.
The $5^2 = 3^2 + (-4)^2$ triple is a variant of the $5^2 = 4^2 + 3^2$ triple calculated when u = 1.
Finally, last triple is the trivial solution $3^2 = 0^2 + (-3)^2$ which just equates a number to itself with the addition of zero.

5.0) Pythagorean Patterns of Behaviour.

There now follows a list of the Pythagorean triples calculated from seed values of 'u' between 1 and 10 to illustrate the patterns that exist in these integers. A Basic programme was written for this task. The two most interesting patterns that emerged quite quickly from this limited study were:

1) Each triple was made from two odd integers and one even integer.
2) The area of the triangle and the sum of the triples were related to the seed integer, 'u'.

Both these topics are discussed later in this paper, but expressed mathematically, it will be shown that:

Seed integer, u = (2 x Triangle Area / Sum of triples) ► u = (BC / (A+B+C))

Using this new fact, it has been possible to calculate the seed integer for the impressive Sumerian and Babylonian triple, A = 3541, B = 2700, C = 2291 was 725. Naturally, they would have been completely unaware of the existence of this seed integer, nor the other triples that this number produced. A full catalogue of the '725' triples is reproduced at the end of this section.

In the following tabulation, each seed integer generates a number of Pythagorean triples, some unique, some variants, but always one trivial triple. For the purposes of clarity, each triple is described by these terms.

u = 1

$5^2 = 4^2 + 3^2$	Unique
$1^2 = 0^2 + (-1)^2$	Trivial

There are two solutions, including the trivial solution is; $u^2 = 0^2 + (-u)^2$

u = 2

$13^2 = 12^2 + 5^2$	Unique
$5^2 = 4^2 + (-3)^2$	Variant
$10^2 = 8^2 + 6^2$	$(5u)^2 = (4u)^2 + (3u)^2$)
$2^2 = 0^2 + (-2)^2$	Trivial

Although there are four solutions, only one is original. The $5^2 = 4^2 + (-3)^2$ triple is a variation of $5^2 = 4^2 + 3^2$ & $10^2 = 8^2 + 6^2$ is $(5u)^2 = (4u)^2 + (3u)^2$. The trivial solution is again $u^2 = 0^2 + (-u)^2$

u = 3

$25^2 = 24^2 + 7^2$	Unique
$17^2 = 15^2 + 8^2$	Unique
$15^2 = 12^2 + 9^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$13^2 = 12^2 + (-5)^2$	Variant
$5^2 = (-4)^2 + 3^2$	Variant
$3^2 = 0^2 + (-3)^2$	Trivial

There are six (2u) solutions, but only two are original. $13^2 = 12^2 + (-5)^2$ and $5^2 = 3^2 + (-4)^2$ are variants of $13^2 = 12^2 + 5^2$ and $5^2 = 4^2 + 3^2$ and $15^2 = 12^2 + 9^2$ is just $(5u)^2 = (4u)^2 + (3u)^2$. Note again the trivial solution is also $u^2 = 0^2 + (-u)^2$

$\underline{u = 4}$

$41^2 = 40^2 + 9^2$	Unique
$25^2 = 24^2 + (-7)^2$	Variant
$26^2 = 24^2 + 10^2$	$2 \times (13^2 = 12^2 + 5^2)$
$10^2 = 8^2 + (-6)^2$	$2 \times (5^2 = 4^2 + (-3)^2)$
$20^2 = 16^2 + 12^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$4^2 = 0^2 + (-4)^2$	Trivial

$\underline{u = 5}$

$61^2 = 60^2 + 11^2$	Unique
$37^2 = 35^2 + 12^2$	Unique
$41^2 = 40^2 + (-9)^2$	Variant
$17^2 = 15^2 + (-8)^2$	Variant
$25^2 = 20^2 + 15^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$5^2 = 0^2 + (-5)^2$	Trivial

$\underline{u = 6}$

$85^2 = 84^2 + 13^2$	Unique
$29^2 = 21^2 + 20^2$	Unique
$61^2 = 60^2 + (-11)^2$	Variant
$5^2 = (-4)^2 + (-3)^2$	Variant
$39^2 = 36^2 + 15^2$	$3 \times (13^2 = 12^2 + 5^2)$
$15^2 = 12^2 + (-9)^2$	$3 \times (5^2 = 4^2 + (-3)^2)$
$50^2 = 48^2 + 14^2$	$2 \times (25^2 = 24^2 + 7^2)$
$26^2 = 24^2 + (-10)^2$	$2 \times (13^2 = 12^2 + (-5)^2)$
$34^2 = 30^2 + 16^2$	$2 \times (17^2 = 15^2 + 8^2)$
$10^2 = (-8)^2 + 6^2$	$2 \times (5^2 = 4^2 + 3^2)$
$30^2 = 24^2 + 18^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$6^2 = 0^2 + (-6)^2$	Trivial

$\underline{u = 7}$

$113^2 = 112^2 + 15^2$	Unique
$65^2 = 63^2 + 16^2$	Unique
$85^2 = 84^2 + (-13)^2$	Variant
$37^2 = 35^2 + (-12)^2$	Variant
$35^2 = 28^2 + 21^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$7^2 = 0^2 + (-7)^2$	Trivial

$\underline{u = 8}$

$145^2 = 144^2 + 17^2$	Unique
$113^2 = 112^2 + (-15)^2$	Variant
$82^2 = 80^2 + 18^2$	$2 \times (40^2 = 40^2 + 9^2)$
$50^2 = 48^2 + (-14)^2$	$2 \times (25^2 = 24^2 + (-7)^2)$
$52^2 = 48^2 + 20^2$	$4 \times (13^2 = 12^2 + 5^2)$
$20^2 = 16^2 + (-12)^2$	$4 \times (5^2 = 4^2 + (-3)^2)$
$40^2 = 32^2 + 24^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$8^2 = 0^2 + (-8)^2$	Trivial

u = 9

$181^2 = 180^2 + 19^2$	Unique
$145^2 = 144^2 + (-17)^2$	Variant
$101^2 = 99^2 + 20^2$	Unique
$65^2 = 63^2 + (-16)^2$	Variant
$75^2 = 72^2 + 21^2$	$3 \times (25^2 = 24^2 + 7^2)$
$39^2 = 36^2 + (-15)^2$	$3 \times (13^2 = 12^2 + (-5)^2)$
$51^2 = 45^2 + 24^2$	$3 \times (17^2 = 15^2 + 8^2)$
$15^2 = (-12)^2 + 9^2$	$3 \times (5^2 = 4^2 + 3^2)$
$45^2 = 36^2 + 27^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$9^2 = 0^2 + (-9)^2$	Trivial

u = 10

$221^2 = 220^2 + 21^2$	Unique
$181^2 = 180^2 + (-19)^2$	Variant
$122^2 = 120^2 + 22^2$	$2 \times (61^2 = 60^2 + 11^2)$
$82^2 = 80^2 + (-18)^2$	$2 \times (41^2 = 40^2 + (-9)^2)$
$74^2 = 70^2 + 24^2$	$2 \times (37^2 = 35^2 + 12^2)$
$34^2 = 30^2 + (-16)^2$	$2 \times (17^2 = 15^2 + (-8)^2)$
$65^2 = 60^2 + 25^2$	$5 \times (13^2 = 12^2 + 5^2)$
$25^2 = 20^2 + (-15)^2$	$5 \times (5^2 = 4^2 + (-3)^2)$
$53^2 = 45^2 + 28^2$	Unique
$13^2 = (-12)^2 + 5^2$	Variant
$50^2 = 40^2 + 30^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$10^2 = 0^2 + (-10)^2$	Trivial

The Sumerian & Babylonian Seed Integer: 725

$1052701^2 = 1052700^2 + 1451^2$	Unique
$527077^2 = 527075^2 + 1452^2$	Unique
$42341^2 = 42340^2 + 291^2$	Unique
$21317^2 = 21315^2 + 292^2$	Unique
$3757^2 = 3132^2 + 2075^2$	Unique
$3625^2 = 2900^2 + 2175^2$	$(5u)^2 = (4u)^2 + (3u)^2$
$3541^2 = 2700^2 + 2291^2$	Sumerian & Babylonian
$1997^2 = 1972^2 + 315^2$	Unique
$1741^2 = 1740^2 + 59^2$	Unique
$1301^2 = 1300^2 + 51^2$	Unique
$1181^2 = 1131^2 + 340^2$	Unique
$901^2 = 899^2 + 60^2$	Unique
$725^2 = 0^2 + (-725)^2$	Trivial
$677^2 = 675^2 + 52^2$	Unique
$61^2 = 60^2 + 11^2$	Variant
$37^2 = 35^2 + 12^2$	Variant
$1049801^2 = 1049800^2 + (-1449)^2$	Variant
$524177^2 = 524175^2 + (-144811)^2$	Variant
$41761^2 = 41760^2 + (-289)^2$	Variant
$20737^2 = 20735^2 + (-288)^2$	Variant
$1625^2 = 1624^2 + (-57)^2$	Variant
$1417^2 = 1392^2 + (-265)^2$	Variant
$1201^2 = 1200^2 + (-49)^2$	Variant

$857^2 = (-825)^2 + 232^2$	Variant
$785^2 = 783^2 + (-56)^2$	Variant
$641^2 = (-609)^2 + (-200)^2$	Variant
$601^2 = 551^2 + (-240)^2$	Variant
$577^2 = 575^2 + (-48)^2$	Variant
$41^2 = 40^2 + (-9)^2$	Variant
$17^2 = 15^2 + (-8)^2$	Variant

6.0) Prime Integer Seeds

Every Pythagorean triple begins with a seed integer 'u', but when that seed integer is a prime number, i.e. a number that only has two factors, itself and unity, this prime number can only produce a maximum of three pairs of factors and hence only six Pythagorean triples. Let P be any prime number seed and because prime numbers are always odd* numbers, by substitution, P = (2m +1), where m is any arbitrary integer.

Since the starting point for any Pythagorean triple is $2u^2$, in this case u = prime number, P

Hence $2P^2 = 2(2m + 1)(2m + 1) = 2(4m^2 + 4m + 1) = (8m^2 + 8m + 2)$

The first pair of factors of $2P^2$ will be unity and the even number $(8m^2 + 8m + 2)$

The second pair of factors will be 2 and the odd number $(4m^2 + 4m + 1)$, found by dividing $2P^2$ by 2.

Note that P is by definition an odd* number, so P^2 must be odd and equal to $(4m^2 + 4m + 1)$.

The third and final pair of factors can only be the original prime number P and an even number equal to 2P. This can be shown as follows:

If P is the original odd prime seed integer, then P = (2m + 1) and $2P^2 = (8m^2 + 8m + 2)$

$$
\begin{array}{rl}
 & \underline{4m + 2} \\
(2m + 1) \mid & (8m^2 + 8m + 2) \\
 & \underline{8m^2 + 4m} \\
 & \quad 4m + 2 \\
 & \quad \underline{4m + 2} \\
 & \quad 0 + 0
\end{array}
$$

This just a long winded way of saying, $2P^2 /P = 2P$, where P = (2m + 1) and 2P = (4m + 2).
In conclusion, when the seed integer is a prime number only three pairs of factors will be produced. These factors will be $2P^2$ and 1, P^2 and 2, 2P and P. Since every factor pair produces two Pythagorean triples, every prime seed integer will only ever produce six Pythagorean triples.

*(It is noted that 2 is a prime even number which does not fit this definition, but u = 2 nonetheless works within this system generating four Pythagorean Triples.)

7.0) Conclusions.

Although not an exhaustive study by any measure, these tables suggest that each seed integer, u:

1) Generates at least one unique Pythagorean Triple.
2) Always generates a trivial set of triples, i.e. $u^2 = 0^2 + (-u)^2$, where 'u' = seed integer.
3) Always generates a $(5u)^2 = (4u)^2 + (3u)^2$ Pythagorean triple, where 'u' = seed integer.
4) Generates variants of earlier triples of the type $A^2 = \pm B^2 \pm C^2$
5) Generates multiple variants of earlier triples of the type $mA^2 = \pm mB^2 \pm mC^2$ where m can be any integer up to, but not exceeding u.
6) Prime number seed integers will never generate more than six Pythagorean triples.
7) For every unique Pythagorean triple, twice the triangle area, (2 x ½BC), divided by the sum of the triples, (A+B+C) always equals the value of the seed integer, u. Mathematically, u = BC/(A+B+C).

It has been also noted that since A, B & C can be positive or negative integers, the magnitude of 'u' will remain, whilst the sign will change. This means that the Pythagorean triples generated by a seed integer 'u' will return ± u when used to back calculate 'u'. Mathematically, u = ±BC/(A+B+C), or Mod |u| = BC/(A+B+C).

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8.0) Prove u = 2*Area / Sum

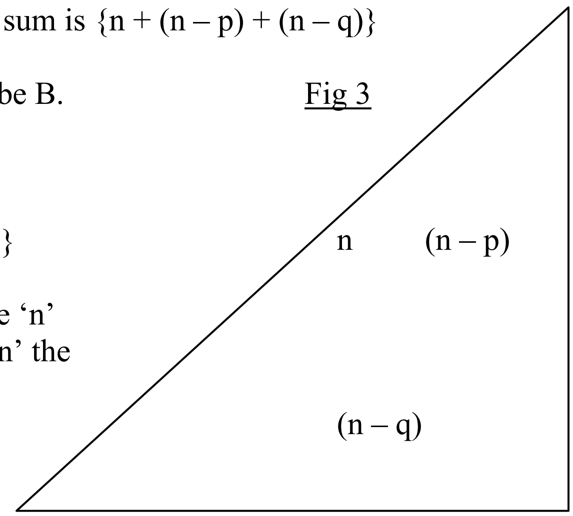
Circumstantial evidence from the Pythagorean triples developed with this technique suggested there was a definite mathematical relationship between the seed integer ‘u’ and some properties of the right angled triangle, i.e. u = twice the area of the triangle divided by the sum of the sides,

Mathematically, $u = 2 \times ½ (n – p)(n – q) / (n + (n – p) + (n – q))$

Where the area of the triangle is $½ \times (n – p)(n – q)$ and the sum is $\{n + (n – p) + (n – q)\}$

Let the area of the triangle be A and the sum of the triples be B.

Fig 3



Then the proposition to be proved is u = 2A / B.

i.e. $2A / B = 2 \times 0.5 \times (n – p)(n – q) / \{n + (n – p) + (n – q)\}$

At this point we note that 2A / B is a constant and therefore ‘n’ cannot appear in the solution. We therefore substitute for ‘n’ the quadratic solution from Eq 3.

i.e. $n_1, n_2 = (p + q) \pm \sqrt{2pq}$

$2A/B = (p+q+\sqrt{2pq} - p)(p+q+\sqrt{2pq} - q) / \{(p+q+\sqrt{2pq}) + (p+q+\sqrt{2pq} – p) + (p+q+\sqrt{2pq} – q)\}$

$2A/B = (q + \sqrt{2pq})(p + \sqrt{2pq}) / (2p + 2q + 3\sqrt{2pq})$

$2A/B = (pq + q\sqrt{2pq} + p\sqrt{2pq} + 2pq) / (2(p + q) + 3\sqrt{2pq})$

$2A/B = (3pq + \sqrt{2pq}(p + q)) / (2(p + q) + 3\sqrt{2pq})$

We can introduce ‘u’ at this point by invoking the earlier relationship $pq = 2u^2$ and substitute.

$2A/B = (3(2u^2) + \sqrt{2(2u^2)}(p + q)) / (2(p + q) + 3\sqrt{2(2u^2)})$

$2A/B = (6u^2 + \sqrt{4u^2}(p + q)) / (2(p + q) + 3\sqrt{4u^2})$ ► $= (6u^2 + 2u(p + q)) / (2(p + q) + 6u)$

$2A/B = u(6u + 2(p + q)) / (2(p + q) + 6u)$ Note that $(6u + 2(p + q)) / (2(p + q) + 6u)$ cancel

Leaving 2A/B = u. QED.

This result proves that for all Pythagorean triples, the seed integer ‘u’ can always be determined from the numerical properties of the triangle and visa versa.

9.0) Evaluating the difference between n_1 and n_2

Given the quadratic solutions are $n_1, n_2 = (p + q) \pm \sqrt{2pq}$ the difference between every associated Pythagorean triple can be calculated.

$n_1 - n_2 = \{(p + q) + \sqrt{2pq}\} - \{(p + q) - \sqrt{2pq}\}$

$n_1 - n_2 = \{(p + q) + \sqrt{2pq} - (p + q) + \sqrt{2pq}\}$

$n_1 - n_2 = \{ 2\sqrt{2pq}\}$ Here we can substitute $2u^2$ for pq

$n_1 - n_2 = \{ 2\sqrt{2(2u^2}\}$

$n_1 - n_2 = \{ 2\sqrt{4u^2}\}$

$n_1 - n_2 = 4u$

This result shows that every pair of Pythagorean triples are separated by a constant equal to four times the seed value used to generate the triples. For example, u = 1 generates the trivial triple, 1, 1, 0 and the unique triple 5, 4, 3. Thus 4u = 4(1) and (5 – 1) = 4.

Similarly, u = 2 generates triples 13, 12, 5 and 5, 4, 3. Thus 4u = 4(2) = 8 = (13 - 5). And so on for all values of u. It is interesting to note that when all the Pythagorean triples are written as an ascending series, the hypotenuse of each triple appears to be separated by its neighbour by either 4 or 8.

10.0) Evaluating 'd' as a multiplier.

We can re-write Eqn 2 with variable, 'd' to signify the quotient p/q to investigate if there are any useful multiplier relationships between the Pythagorean triples.

From $n^2 = (n - p)^2 + (n - q)^2$ (Eqn 2)

Let $d = p/q$ ► $p = qd$

Then: $n^2 = (n - qd)^2 + (n - q)^2$

$n^2 = (n - qd)(n - qd) + (n - q)(n - q)$ ► $n^2 = n^2 - 2nqd + (qd)^2 + n^2 - 2nq + q^2$

Collecting terms, $n^2 = 2n^2 - 2n(qd + q) + (qd)^2 + q^2$

$$n^2 - 2nq(1 + d) + q^2(1 + d^2) = 0$$

Using the standard quadratic formula:

$$n_1, n_2 = 2q(1 + d) \pm\sqrt{(-2q(1 + d))^2 - 4(1)(q^2(1 + d^2)} / 2(1)$$

$$n_1, n_2 = 2q(1 + d) \pm\sqrt{4q^2(1 + 2d + d^2) - 4q^2(1 + d^2)} / 2$$

$$n_1, n_2 = 2q(1 + d) \pm\sqrt{(4q^2 + 8dq^2 + 4q^2d^2 - 4q^2 - 4q^2d^2)} / 2$$

$n_1, n_2 = 2q(1 + d) \pm\sqrt{8dq^2} / 2$ ► $n_1, n_2 = 2q(1 + d) \pm\sqrt{4q^2(2d)} / 2$

$$n_1, n_2 = 2q(1 + d) \pm 2q\sqrt{2d} / 2 \quad \blacktriangleright \quad n_1, n_2 = q\{(1 + d) \pm \sqrt{2d}\}$$

With this solution, it will be noted that 'q' has been reduced to a simple multiplier and will not affect the results, other than to scale each answer. Therefore for the purpose of continuing this analysis, we let q = 1. The surd, $\pm\sqrt{2d}$ limits the useful range of integer solutions and we know from similar analysis on page 5 that 'd' cannot be odd and therefore must be even.

Let $2u = \pm\sqrt{2d}$, then $4u^2 = 2d$, i.e. $d = 2u^2$ where 'u' is any positive, zero or negative seed integer.

Integer values of n_1, n_2 can be calculated from a seed integer 'u'. If we plot the analysis with q = 1, it produces a continuous series of numbers which are recognised as the hypotenuse or largest numbers of each unique Pythagorean triple. Table 3 shows the data for the first twenty nine integer values of 'u'.

Table 3: Ascending Order of Pythagorean Triples based upon a seed integer 'u'. (q = 1)

'u'	$d = 2u^2$	2d	$(1 + d) \pm\sqrt{2d}$	n_1, n_2	$4u = (n_1 - n_2)$
0	0	0	1 ± 0	1 or 1	0
1	2	4	3 ± 2	5 or 1	4
2	8	16	9 ± 4	13 or 5	8
3	18	36	19 ± 6	25 or 13	12
4	32	64	33 ± 8	41 or 25	16
5	50	100	51 ± 10	61 or 41	20
6	72	144	73 ± 12	85 or 61	24
7	98	196	99 ± 14	113 or 85	28
8	128	256	129 ± 16	145 or 113	32
9	162	324	163 ± 18	181 or 145	36
10	200	400	201 ± 20	221 or 181	40
11	242	484	243 ± 22	265 or 221	44
12	288	576	289 ± 24	313 or 265	48
13	338	676	339 ± 26	365 or 313	52
14	392	784	393 ± 28	421 or 365	56
15	450	900	451 ± 30	481 or 421	60
16	512	1024	513 ± 32	545 or 481	64
17	578	1156	579 ± 34	613 or 545	68
18	648	1296	649 ± 36	685 or 613	72
19	722	1444	723 ± 38	761 or 685	76
20	800	1600	801 ± 40	841 or 761	80
21	882	1764	883 ± 42	925 or 841	84
22	968	1936	969 ± 44	1013 or 925	88
23	1058	2116	1059 ± 46	1105 or 1013	92
24	1152	2304	1153 ± 48	1201 or 1105	96
25	1250	2500	1251 ± 50	1301 or 1201	100
26	1352	2704	1353 ± 52	1405 or 1301	104
27	1458	2916	1459 ± 54	1513 or 1405	108
28	1568	3136	1569 ± 56	1625 or 1513	112
29	1682	3364	1683 ± 58	1741 or 1625	116

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Close analysis of this series reveals that this technique is imperfect because several well known Pythagorean triples are missing from the list. For example, 17, 15, 8 and 29, 21, 20 and 37, 35, 12 and 65, 63, 16 to name just the first four. This is due to the method being too simple and its failure to recognise all the p and q factors of the seed, $2u^2$. It's clear that only the first two factors are being considered, i.e. 1 and $2u^2$ which generates the leading numbers from the two largest Pythagorean triples. Unfortunately, because the method obscures any remaining p and q factors, it therefore fails to generate any remaining Pythagorean triples that could be calculated from these alternative factors of $2u^2$. This failure of the mathematics is true, despite the series appearing to satisfy the earlier statement that the difference between n_1 and n_2 is always four times the seed integer, i.e. $(n_1 - n_2) = 4u$ and despite the fact that the missing triples are separated by their nearest neighbours by either 4 or 8. This data set also shows that the method includes redundancy, because the largest integer calculated from any value of 'u', becomes the smallest integer when 'u' is incremented to 'u + 1'. Finally, to illustrate the over all shape of these curves and to show the relationship between adjacent Pythagorean quadratics, the series from u = 1 to u = 15 has been plotted in Fig 3.

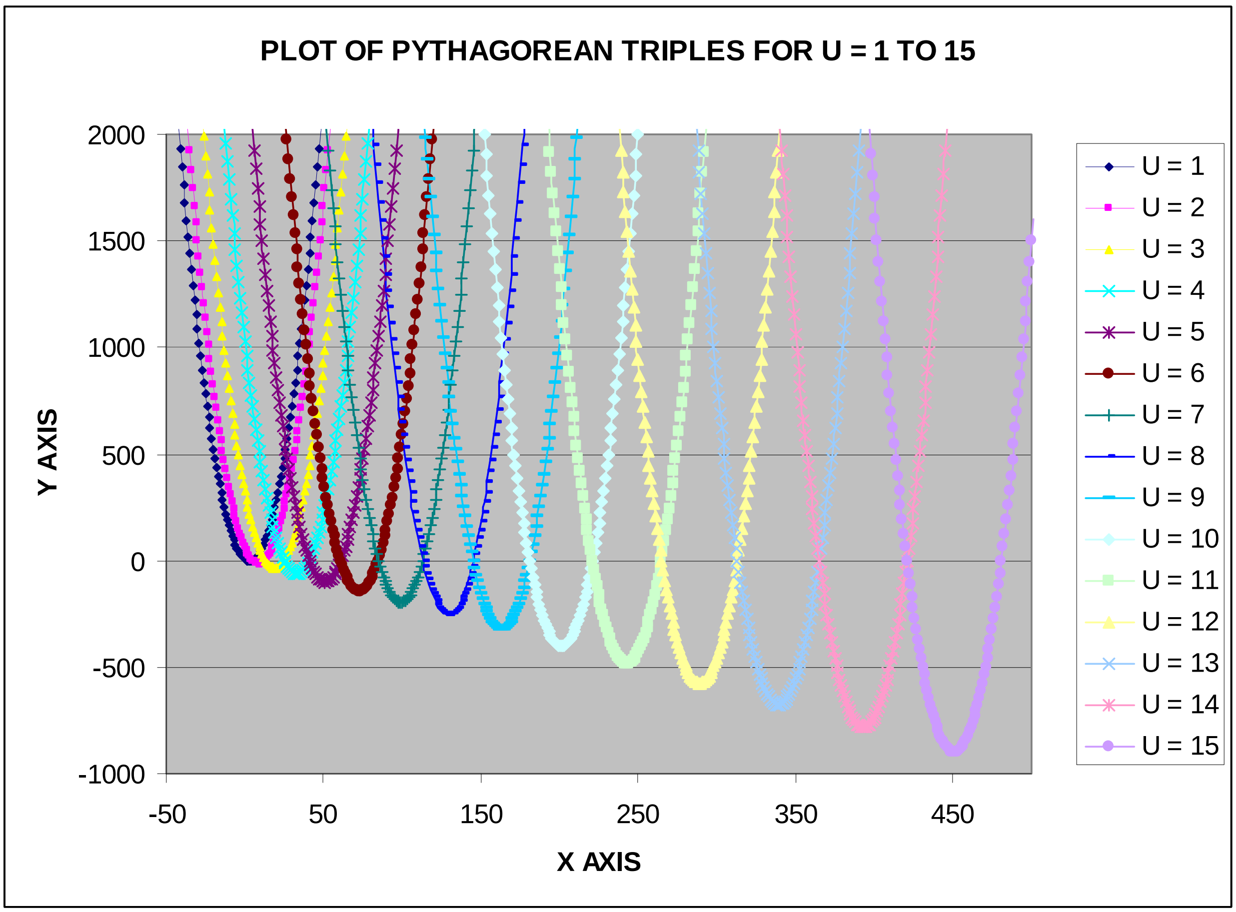

Fig 4: Plot of Pythagorean triples for u = 1 to 15.

This set of curves were derived from individual quadratic equations, themselves derived from a seed integer 'u' and the first two factors of $2u^2$, i.e. 1 and $2u^2$. For example the first curve used u =1 for the seed integer, from which were derived the first two factors, 1 and $2(1)^2$, i.e. 1 and 2. These integers were substituted in the equation:

$$n^2 + 2n(p + q) + (p^2 + q^2) = 0 \text{..........(Eq 2)}$$

$$n^2 + 2n(1 + 2) + (1^2 + 2^2) = 0$$

$$n^2 + 6n + 5 = 0$$

This equation and all the other equations up to u = 15 were calculated and plotted in Fig 3.

11.0) Evaluating the difference between p and q.

We can re-write Eqn 2 with another variable, 'd' to signify the difference between p and q.

$$\text{From} \quad n^2 = (n - p)^2 + (n - q)^2 \qquad \text{(Eqn 2)}$$

$$\text{Let } d = (p - q) \quad \blacktriangleright \quad p = d + q$$

$$\text{Then:} \quad n^2 = (n - (d + q))^2 + (n - q)^2$$

$$n^2 = (n - (d + q))(n - (d + q)) + (n - q)(n - q)$$

$$n^2 = n^2 - n(d + q) - n(d + q) + (d + q)^2 + n^2 - nq - nq + q^2$$

Collecting terms, $n^2 = 2n^2 - n\{2(d + q) + 2q\} + \{(d + q)^2 + q^2\}$

$$n^2 - 2n\{(d + q) + q\} + \{(d + q)^2 + q^2\} = 0$$

$$n^2 - 2n(d + 2q) + (d^2 + 2dq + 2q^2) = 0$$

Using the standard quadratic formula:

$$n_1, n_2 = 2(d + 2q) \pm\sqrt{(2(d + 2q))^2 - 4(1)(d^2 + 2dq + 2q^2)} / 2(1)$$

$$n_1, n_2 = 2(d + 2q) \pm\sqrt{4(d^2 + 4dq + 4q^2) - 4d^2 - 8dq - 8q^2} / 2$$

$$n_1, n_2 = 2(d + 2q) \pm\sqrt{4d^2 + 16dq + 16q^2 - 4d^2 - 8dq - 8q^2} / 2$$

$$n_1, n_2 = 2(d + 2q) \pm\sqrt{8dq + 8q^2} / 2$$

$$n_1, n_2 = 2(d + 2q) \pm\sqrt{4q(2d + 2q)} / 2$$

$$n_1, n_2 = 2(d + 2q) \pm 2\sqrt{q(2d + 2q)} / 2$$

$$n_1, n_2 = (d + 2q) \pm\sqrt{2q(d + q)}$$

Using this result, we can test what values of 'd', i.e. what values of (p – q) will affect the creation of Pythagorean triples. For example;

11.1) When d = 0

$$n_1, n_2 = (d + 2q) \pm\sqrt{2q(d + q)}$$

$$n_1, n_2 = (0 + 2q) \pm\sqrt{2q(0 + q)}$$

$$n_1, n_2 = 2q \pm\sqrt{2q^2}$$

$$\text{i.e. } n_1, n_2 = 2q \pm q\sqrt{2}$$

Therefore

$$n_1 - n_2 = (2q + q\sqrt{2}) – (2q - q\sqrt{2})$$

$$n_1 - n_2 = 2q + q\sqrt{2} – 2q + q\sqrt{2})$$

$$n_1 - n_2 = 2q\sqrt{2}$$

This result equates the irrational number, $\sqrt{2}$ with n_1, n_2 and q which are by definition integers. Therefore there are no integer solutions when d = 0, i.e. when (p – q) = 0.

11.2) When d = 1 or higher.

This analytical process can be extended for any integer value of d and when d = 1, this equation does successfully predict the 5, 4, 3 Pythagorean triangle as expected. But other than proving that some integer values for d will not produce any Pythagorean triples, it's hard to see how this line of mathematical enquiry will lead to any valuable conclusions, over and above what has already been discovered.

12.0) More Properties of Pythagorean Triples.

12.1) Proof that $b \neq c$.

When considering the potential integer solutions for the equation $a^2 = b^2 + c^2$, we must consider the conditions for a, b and c to exist. For example, logic defines $a > b + c$, but can $c = b$?.

If $c = b$, then we can write $a^2 = b^2 + b^2$ ► $a^2 = 2b^2$

Therefore $a = b\pm\sqrt{2}$. But 'a' is defined as an integer and $\sqrt{2}$ is an irrational number. So regardless of the integer value of 'b', we're left with the conclusion that for the equation $a^2 = b^2 + c^2$ to have integer solutions, 'b' can never equal 'c'.

12.2) Odd or Even Values for a, b and c.

Close analysis of the Pythagorean triples shows that they always appear in terms of two odd numbers and one even number.

Let a, b and c exist as integer solution for the equation: $a^2 = b^2 + c^2$

Therefore 'a' must be greater than 'b' and 'a' must be greater than 'c', i.e. $a > b$ and $a > c$
As proved in section 12.1, $b \neq c$, so b must also be greater than c, i.e. $b > c$.

So we're now able to write that as a necessary condition for the equation $a^2 = b^2 + c^2$ to have integer solutions, $a > b > c$.

12.2.1) All Odd Integers.

If a, b and c exist as odd integer solutions of $a^2 = b^2 + c^2$, then by definition, a, b and c cannot be reduced further by dividing by 2. However the following analysis proves a, b and c cannot all be odd integers.

If a, b and c are all odd numbers, then a^2, b^2 and c^2 are also odd.

12.2.2) Proof

If 'a' be a generalised odd integer equal to $(2n + 1)$, then $(2n + 1)^2$ is also odd because the remainder is equal to 1.

Therefore if a, b and c are odd integer solutions of $a^2 = b^2 + c^2$, then

$$(2n +1)^2 = (2p + 1)^2 + (2q +1)^2$$

$$(2n +1)^2 = (4p^2 + 4p + 1) + (4q^2 +4q + 1)$$

$$(2n +1)^2 = 4(p^2 + q^2) + 4(p + q) + 2$$

Regardless of the exact values for p and q, the right hand side of this equation is always even, whereas the left hand side is always odd. Therefore it's impossible for the solutions of the equation $a^2 = b^2 + c^2$, to be all odd integers.

12.2.3) All Even Integers.
If a, b and c exist as even integer solutions of $a^2 = b^2 + c^2$, then by definition, a, b and c represent a scalar solution because these integers can be reduced further by dividing by 2. Therefore division by 2 should continue until one of the integers becomes odd. This statement is reiterated in part 1) of section 12.2.5

12.2.4) Some Even, Some Odd Integers.
If a, b and c exist as a mixture of odd and even integer solutions of $a^2 = b^2 + c^2$, then by definition, a, b and c represent a fundamental solution because at least one of the terms cannot be reduced further by dividing by 2.

12.2.5) Proof of one even and two odd integers.
1) If a, b and c exist as integer solutions of $a^2 = b^2 + c^2$, then by definition, a, b and c cannot all be even numbers because the set could be continuously divided by 2 until at least one number became odd. Therefore every fundamental Pythagorean triple must contain at least one odd integer.

2) Solutions a, b and c cannot all be odd integers because adding two odd numbers produces an even number, which contradicts 'a' being and odd integer. Therefore at least one even integer must exist.

3) If 'b' and 'c' are both even, then b^2 and c^2 are also both even, which means a^2 must also be even. However, this conclusion contradicts 1) above, therefore 'b' or 'c' must be odd. If this statement is true, then 'b' or 'c' must be even and therefore 'a' must always be odd.

4) Logic dictates that all Pythagorean triples consist of one even and two odd integers. Integer 'a' will always be odd, which forces 'b' or 'c' to be the remaining odd integer. If 'b' is odd, then 'c' must be even or visa versa.

12.2.6) Conclusion
All fundamental solutions for the equation $a^2 = b^2 + c^2$, must consist of one even and two odd integers. Integer 'a' will always be odd, which forces 'b' or 'c' to be the remaining odd integer. If 'b' is odd, then 'c' must be even or visa versa.

13.0) Problems with Higher Order Solutions

This paper has so far described a method for calculating Pythagorean triples from a seed integer 'u' using polynomials developed from the sides of a right angled triangle. Fermat's assertion was that given the general equation; $a^n = b^n + c^n$, there were no integer solutions for all values of 'n' greater than 2. Although not a proof, this method of polynomial analysis supports this assertion using the following logical argument.

When n = 2, it created a second order equation whose solution was; $n_1, n_2 = (p + q) \pm \sqrt{2pq}$, i.e. two solutions from three coefficients in two variables.

When n = 3, a solution would be required which produced three values of n, i.e. n_1, n_2 and n_3 from four coefficients but using the same two variables p and q. For example, one could imagine the solutions taking the form;

$$n_1 = (p + q),\ n_{2,} = (p + q) + \sqrt{6pq},\ n_3 = (p + q) - \sqrt{6pq}$$

or

$$n_1 = (p + q),\ n_{2,} = (p + q) + \sqrt{3pq},\ n_3 = (p + q) - \sqrt{3pq}$$

both examples very nearly satisfying the cubic equation, so three solutions in 'n' from two variables is difficult, but not impossible.

When n = 4, a solution would now be required which produced four values of n, i.e. n_1, $n_{2,}$ n_3 and n_4 from the same two variables p and q, which would be even more difficult than the cubic to imagine because we're running out of ways to arrange the two variables, p and q. A fourth order solution could be in the form of:

$$n_1 = (p + q) + \sqrt{2pq},\ n_2 = (p + q) - \sqrt{2pq},\ n_3 = (p - q) + \sqrt{2pq},\ n_4 = (p - q) - \sqrt{2pq}$$

but we're very quickly running out combinations and permutations of p and q and 5^{th}, 6^{th} and higher integer solutions look less and less likely. In the limit, when required to find integer solutions to the equation $a^n = b^n + c^n$, when $n = \infty$, it's obvious that there is no solution because an infinite combination of p and q would be required, which simply does not exist. Therefore, because there are only two free variables, (p + q) to determine integer solutions for $a^n = b^n + c^n$, for any value of 'n', there are unlikely to be any integer solutions when n > 2 because we only have two free variables, p and q in the expanded polynomial equation.

14.0) Incremental analysis applied to the first four orders of the general equation, $a^n = b^n + c^n$.
Incremental analysis is a very useful way to determine if an unknown solution can exist. The process assumes there is a linear relationship between all solutions and attempts to prove it exists. If a linear relationship can be shown not to exist, this negative result suggests no solution exists. The following three examples demonstrate the technique and a graph of each solution is presented in Fig 5 to reinforce the conclusions drawn from this analysis.

14.1) The linear case when n = 1.
When n = 1, the general equation is reduced to the familiar straight line equation, a = b + c.

We assume solutions exist to satisfy this equation such that another, slightly larger solution exists.

$$\text{So that: } (a + \delta a) = (b + \delta b) + (c + \delta c)$$

$$\text{Re-arranging, we get: } (a - b - c) = \delta b + \delta c - \delta a$$

$$\text{If a solution to } a = b + c \text{ exists, then } (a - b - c) = 0 \text{ and hence } (\delta b + \delta c - \delta a) = 0$$

Let δ be some constant of proportionality 'k', such that ka = δa, kb = δb and kc = δc

$$\text{Then} \quad kb + kc - ka = 0 \qquad \text{and} \qquad ka = kb + kc$$

Since 'k' is any arbitrary constant of proportionality, the next or previous solution of the equation a = b + c is simply the next, or previous multiple, ak = bk + ck. The infinite solutions for (a = b + c) are characterised by a straight line graph that has equal positive and negative solutions and passes through zero.

14.2) The quadratic case when n = 2.
When n = 2, the general equation is reduced to the familiar quadratic equation, $a^2 = b^2 + c^2$.

We assume solutions exist to satisfy this equation such that another, slightly larger solution exists.

$$\text{So that: } (a + \delta a)^2 = (b + \delta b)^2 + (c + \delta c)^2$$

$$\text{Re-arranging, we get: } (a^2 + 2a\delta a + \delta a^2) = (b^2 + 2b\delta b + \delta b^2) + (c^2 + 2c\delta c + \delta c^2)$$

$$\text{Therefore } (a^2 - b^2 - c^2) = (2b\delta b + \delta b^2 + 2c\delta c + \delta c^2 - 2a\delta a - \delta a^2)$$

$$\text{Again, if a solution for } a^2 = b^2 + c^2 \text{ exists, then } a^2 - b^2 - c^2 = 0$$

$$\text{And therefore } (2b\delta b + \delta b^2 + 2c\delta c + \delta c^2 - 2a\delta a - \delta a^2) = 0$$

$$\delta b(2b + \delta b) + \delta c(2c + \delta c) - \delta a(2a + \delta a) = 0$$

Let δ be some constant of proportionality such that ka = δa, kb = δb and kc = δc

$$\text{Then} \qquad kb(2b + kb) + kc(2c + kc) - ka(2a + ka) = 0$$

$$b(2b + kb) + c(2c + kc) - a(2a + ka) = 0$$

$$b^2(2 + k) + c^2(2 + k) – a^2(2 + k) = 0$$

Re-written: $$a^2(2 + k) = b^2(2 + k) + c^2(2 + k)$$

Since ‘k’ is any arbitrary constant of proportionality, the next or previous solution of the equation $a^2(2 + k) = b^2(2 + k) + c^2(2 + k)$ is simply the next, or previous multiple of k. This solution differs from the previous case when n = 1, by an offset of 2, but nonetheless remains a linear function of hypothetical solutions characterised by a straight line graph that passes through zero when k = -2 with positive and negative solutions. This analysis suggests that simple scalar solutions for the equation $a^2 = b^2 + c^2$ exist.

<u>14.3) The cubic case when n = 3.</u>

When n = 3, the general equation is reduced to the cubic equation, $a^3 = b^3 + c^3$.

We assume solutions exist to satisfy this equation such that another, slightly larger solution exists.

So that: $$(a + \delta a)^3 = (b + \delta b)^3 + (c + \delta c)^3$$

Re-arranging: $$(a^3 + 3a^2\delta a + 3a\delta a^2 + \delta a^3) = (b^3 + 3b^2\delta b + 3b\delta b^2 + \delta b^3) + (c^3 + 3c^2\delta c + 3c\delta c^2 + \delta c^3)$$

$$(a^3 – b^3 – c^3) = (3b^2\delta b + 3b\delta b^2 + \delta b^3 + 3c^2\delta c + 3c\delta c^2 + \delta c^3 – 3a^2\delta a – 3a\delta a^2 – \delta a^3)$$

Again, if a solution for $a^3 = b^3 + c^3$ exists, then $a^3 – b^3 – c^3 = 0$

And $$(3b^2\delta b + 3b\delta b^2 + \delta b^3 + 3c^2\delta c + 3c\delta c^2 + \delta c^3 – 3a^2\delta a – 3a\delta a^2 – \delta a^3) = 0$$

$$\delta b(3b^2 + 3b\delta b + \delta b^2) + \delta c(3c^2 + 3c\delta c + \delta c^2) – \delta a(3a^2 + 3a\delta a + \delta a^2) = 0$$

Let δ be some constant of proportionality such that ka = δa, kb = δb and kc = δc

Then: $$kb(3b^2 + 3bkb + (kb)^2) + kc(3c^2 + 3ckc + (kc)^2) – ka(3a^2 + 3aka + (ka)^2) = 0$$

$$b(3b^2 + 3kb^2 + k^2b^2) + c(3c^2 + 3kc^2 + k^2c^2) – a(3a^2 + 3ka^2 + k^2a^2) = 0$$

$$b^3(3 + 3k + k^2) + c^3(3 + 3k + k^2) – a^3(3 + 3k + k^2) = 0$$

Re-arranging: $$a^3(3 + 3k + k^2) = b^3(3 + 3k + k^2) + c^3(3 + 3k + k^2)$$

This result shows that the cubic is not linearly related to its next, or previous theoretical solution because the scalar, ‘k’ is a quadratic function. Further, this quadratic has complex roots which prevent the curve from passing through zero. As Fig 4 illustrates, this solution does not have any of the linear properties demonstrated for the previous two examples. The result is a complex quadratic curve which does not pass through the origin and does not offer equal positive and negative solutions. Many values of ‘k’ produce the same answer which is not only non linear, but also illogical. For example, the scalar quantity, $(3 + 3k + k^2)$ is +1 when k = -1 or -2 and +3 when k = 0 or -3. This result violates the rule of scalar proportionality upon which this analysis was based and is why this result suggests there are no integer solutions for the cubic equation.

14.4) The quartic case when n = 4.
When n = 4, the general equation is reduced to the quartic equation, $a^4 = b^4 + c^4$.

We assume solutions exist to satisfy this equation such that another, slightly larger solution exists.

$$\text{So that: } (a + \delta a)^4 = (b + \delta b)^4 + (c + \delta c)^4$$

$$(a^4+4a^3\delta a+6a^2\delta a^2+4a\delta a^3+\delta a^4) = (b^4+4b^3\delta b+6b^2\delta b^2+4b\delta b^3+\delta b^4) + (c^4+4c^3\delta c+6c^2\delta c^2+4c\delta c^3+\delta c^4)$$

$$(a^4 - b^4 - c^4) = (4b^3\delta b+6b^2\delta b^2+4b\delta b^3+\delta b^4 + 4c^3\delta c+6c^2\delta c^2+4c\delta c^3+\delta c^4 - 4a^3\delta a - 6a^2\delta a^2 - 4a\delta a^3 - \delta a^4)$$

Again, if a solution for $a^4 = b^4 + c^4$ exists, then $a^4 - b^4 - c^4 = 0$ and

$$(4b^3\delta b+6b^2\delta b^2+4b\delta b^3+\delta b^4 + 4c^3\delta c+6c^2\delta c^2+4c\delta c^3+\delta c^4 - 4a^3\delta a - 6a^2\delta a^2 - 4a\delta a^3 - \delta a^4) = 0$$

$$\delta b(4b^3 + 6b^2\delta b+4b\delta b^2+\delta b^3) + \delta c(4c^3 + 6c^2\delta c + 4c\delta c^2 + \delta c^3) - \delta a(4a^3 + 6a^2\delta a + 4a\delta a^2 + \delta a^3) = 0$$

Let δ be some constant of proportionality such that $ka = \delta a$, $kb = \delta b$ and $kc = \delta c$

$$kb(4b^3 + 6b^2kb+4bkb^2+kb^3) + kc(4c^3 + 6c^2kc + 4ckc^2+kc^3) - ka(4a^3 + 6a^2ka + 4aka^2 + ka^3) = 0$$

$$b(4b^3 + 6b^2kb+4bkb^2+kb^3) + c(4c^3 + 6c^2kc + 4ckc^2+kc^3) - a(4a^3 + 6a^2ka + 4aka^2 + ka^3) = 0$$

$$b^4(4 + 6k + 4k^2 + k^3) + c^4(4 + 6k + 4k^2 + k^3) - a^4(4 + 6k + 4k^2 + k^3) = 0$$

$$\text{Re-arranging: } a^4(4 + 6k + 4k^2 + k^3) = b^4(4 + 6k + 4k^2 + k^3) + c^4(4 + 6k + 4k^2 + k^3)$$

As expected, this result shows that the quartic is also not linearly related to its next, or previous theoretical solution because the scalar, 'k' is a cubic function. Further, this cubic has one real root (when $k = -2$) and two complex roots (when $k = -1 \pm j$). As Fig 4 illustrates, this solution does not have any of the linear properties necessary to suggest the quartic has any linearly related solutions, suggesting there are none.

This technique could be used for higher powers of the general equation $a^n = b^n + c^n$, but the linear relationship between any two theoretical integer solutions does not exist for all equations above the quadratic, i.e. for all values of $n > 2$. Whilst this conclusion is not necessarily a proof, increment analysis applied to Fermat's general equation, $a^n = b^n + c^n$, suggests there are no integer solutions of this equation when $n > 2$. This level of analysis was well within Fermat's knowledge and capabilities and it's possible that this is Fermat's "…..truly marvellous demonstration which this margin is too narrow to contain".

Fig 5 illustrates the curves from this incremental analysis which show the abrupt change from a linear profile for the first and second order equations to a non linear profile for third order and above.

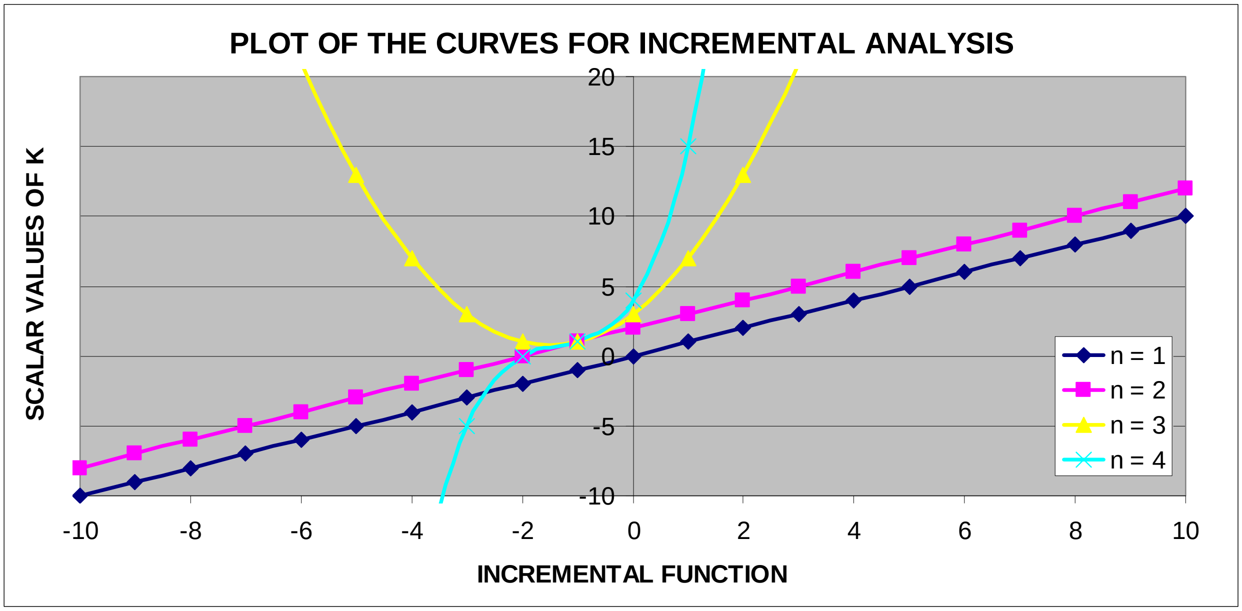

Fig 5: Plot of incremental analysis showing the change in linearity when n > 2

15.0) Expansion of Pythagorean Equations and Pascal's Triangle.

Equations 1 and 2 can be expressed in any order and the expansion developed. For example the general form of the Pythagorean equation is:

$$a^n = b^n + c^n \qquad \text{Eq 1}$$

The modified form is: $n^m = (n - p)^m + (n - q)^m$ (Eqn 2), where m is any integer exponent.

From which we get a series of polynomial expansions for increasing power of 'm' as follows:

$$n^1 - (p + q) = 0$$

$$n^2 - 2n(p + q) + (p^2 + q^2) = 0$$

$$n^3 - 3n^2(p + q) + 3n(p^2 + q^2) - (p^3 + q^3) = 0$$

$$n^4 - 4n^3(p + q) + 6n^2(p^2 + q^2) - 4n(p^3 + q^3) + (p^4 + q^4) = 0$$

$$n^5 - 5n^4(p + q) + 10n^3(p^2 + q^2) - 10n^2(p^3 + q^3) + 5n(p^4 + q^4) - (p^5 + q^5) = 0$$

$$n^6 - 6n^5(p + q) + 15n^4(p^2 + q^2) - 20n^3(p^3 + q^3) + 15n^2(p^4 + q^4) - 6n(p^5 + q^5) + (p^6 + q^6) = 0$$

Expansion of this generalised equation for the first few orders shows that the numerical coefficients follow those predicted by the Binomial Theorem, coefficients commonly referred to a Pascal's triangle due to the work of Blaisé Pascal (1623 – 1662) a contemporary of Pierre de Fermat and who is credited with Fermat for inventing the mathematics of Probability.

15.1) Pascal's Triangle for the first six polynomial coefficients.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Table 4: Pascal's Triangle up to n = 6

15.2) The Binomial Theorem applied to Pythagorean Triples.
Examination of the first six Pythagorean polynomials and Pascal's triangle suggests that the analytical approach proposed by this paper produces Pythagorean polynomials which can be described by the Binomial Theorem, where any polynomial of this type can be generalised by the Binomial Equation:

Appendix 1.0) Fermat's Prime Number Analysis.
This section has no direct relation to the body of the text, but is included for interest.

Fermat showed that every prime integer P, can be expressed as the difference between any two square integers. If P and 1 are the only two factors of a prime integer P, then two other integers, x and y can be shown to equate to P as follows:

If 1 = (x – y), then P = (x + y) and therefore P x 1 = (x – y)(x + y)

Therefore prime integer, P = $(x^2 – y^2)$, i.e. any prime integer can be expressed by the difference between any two integer squares.

By reverse analysis, Fermat also showed that if the prime P, is known, then x and y can be calculated as follows:

P = (x + y) and 1 = (x – y). Therefore by subtraction

P = (x + y)	&	P = (x + y)
1 = (x – y)	&	1 = (x – y)
P – 1 = 2y	&	P + 1 = 2x

i.e. y = (P – 1)/2 and x = (P + 1)/2

Appendix 2.0) Algebraic Analysis of the Cubic Equation
For the cubic equation, $a^3 = b^3 + c^3$, the modified form is:

$$n^3 - 3n^2(p + q) + 3n(p^2 + q^2) - (p^3 + q^3) = 0 \qquad \text{(Eq 4)}$$

which can either have three real roots or one real and two complex roots. Three real roots are defined as places where the cubic equation crosses the 'n' axis. Complex roots are defined as having both a real and an imaginary component (Re ± jIm) and whilst they describe the roots of an equation just as accurately as a real root, they generally point to inflexions in the curve of the equation which do not cross the 'n' axis as opposed to places where the equation crosses the real horizontal 'n' axis. If the cubic has three real roots, it will cross the 'n' axis three times. If the cubic has one real and two complex roots, it will cross the 'n' axis once.

To solve this equation means to find the real and /or complex roots of this equation and either prove or disprove the existence of integer solutions, just as the quadratic equation $n^2 - 2n(p + q) + (p^2 + q^2) = 0$ was solved to produce roots, $n_1, n_2 = (p + q) \pm\sqrt{2pq}$. The first task is therefore to find what type of roots make up this cubic equation.

Appendix 2.1) Positive or Negative Roots.
Let the solution to the cubic be three real roots α, β and γ and let them be either positive or negative real roots, such that the positive solutions are: $n_1 = \alpha$, $n_2 = \beta$, $n_3 = \gamma$,

so that $(n_1 - \alpha) = 0$, $(n_2 - \beta) = 0$, $(n_3 - \gamma) = 0$.

The alternate solution is: $n_1 = -\alpha$, $n_2 = -\beta$, $n_1 = \alpha$, $n_3 = -\gamma$,
so that so that $(n_1 + \alpha) = 0$, $(n_2 + \beta) = 0$, $(n_3 + \gamma) = 0$.

This leads to two different expressions for the cubic:

1) $(n_1 - \alpha)(n_2 - \beta)(n_3 - \gamma) = 0$ and 2) $(n_1 + \alpha)(n_2 + \beta)(n_3 + \gamma) = 0$.

Expanding equation 1

Expanding: $n^3 - n^2(\alpha + \beta + \gamma) + n(\alpha\beta + \alpha\gamma + \beta\gamma) - \alpha\beta\gamma = 0$

Expanding equation 2

Expanding: $n^3 + n^2(\alpha + \beta + \gamma) + n(\alpha\beta + \alpha\gamma + \beta\gamma) + \alpha\beta\gamma = 0$

It is noted that the signs are all positive in the expansion of equation 2, whereas the signs in equation 1 alternate in the same way as the original Eq 3. This result suggests there are no negative roots to the cubic and in fact all the roots are positive and occupy the right hand side of a graph describing the cubic.

Another way to demonstrate this fact is to substitute a positive and negative real root and again look for changes in the sign of the coefficients. For Eq 3, let a positive real root exist, R, such that $(n - R) = 0$,

$$\text{Then} \qquad R^3 - 3R^2(p + q) + 3R(p^2 + q^2) - (p^3 + q^3) = 0$$

The signs of the coefficients remain unchanged, i.e. they still alternate suggesting at least one positive real root exits. Now let a negative real root exist, $-R$, such that $(n + R) = 0$, Then $-R^3 - 3R^2(p + q) - 3R(p^2 + q^2) - (p^3 + q^3) = 0$

This time all the coefficients have the same sign, so no negative roots exist. All three roots are positive.

Appendix 2.2) Real or Complex Roots.

Every cubic has at least one real root, so the question is whether the two remaining roots are real or complex. This analysis can be helped by graphical representation of the cubic and its differentials.

Given $$n^3 - 3n^2(p + q) + 3n(p^2 + q^2) - (p^3 + q^3) = 0 \quad \text{(Eq 3)}$$

Then $$d/dn = 3n^2 - 6n(p + q) + 3(p^2 + q^2) = 0 \quad \text{(Eq 4)}$$

And $$d^2/dn^2 = 6n - 6(p + q) = 0 \quad \text{(Eq 5)}$$

It will be noted that Eq 4 is just 3 times Eq 1: i.e. $3(n^2 - 2n(p + q) + (p^2 + q^2)) = 0$

And therefore the roots of this equation are: $n_1, n_2 = (p + q) \pm\sqrt{2pq}$.

Appendix 2.3) Analysis of the general cubic equation

The general form of the cubic equation is: $n^3 - n^2(\alpha + \beta + \gamma) + n(\alpha\beta + \alpha\gamma + \beta\gamma) - \alpha\beta\gamma = 0$

Differentiating; $d/dn = 3n^2 - 2n(\alpha + \beta + \gamma) + (\alpha\beta + \alpha\gamma + \beta\gamma) = 0$

So it would appear that the roots of the cubic, (α, β, γ) are recorded in the quadratic.

Using the quadratic formula to solve this equation gives:

$$n_1, n_2 = \frac{-(-2(\alpha + \beta + \gamma) \pm\sqrt{(-2(\alpha + \beta + \gamma))^2 - 4(3)(\alpha\beta + \alpha\gamma + \beta\gamma)}}{2(3)}$$

$$n_1, n_2 = \frac{2(\alpha + \beta + \gamma) \pm\sqrt{4(\alpha + \beta + \gamma)^2 - 12(\alpha\beta + \alpha\gamma + \beta\gamma)}}{6}$$

$$n_1, n_2 = \frac{2(\alpha + \beta + \gamma) \pm\sqrt{4((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma))}}{6}$$

$$n_1, n_2 = \frac{2(\alpha + \beta + \gamma) \pm 2\sqrt{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma)}}{6}$$

$$n_1, n_2 = \frac{(\alpha + \beta + \gamma) \pm\sqrt{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma)}}{3}$$

From the cubic equation, we note that $3(p + q) = (\alpha + \beta + \gamma)$ and $3(p^2 + q^2) = (\alpha\beta + \alpha\gamma + \beta\gamma)$

$$n_1, n_2 = \frac{3(p + q) \pm\sqrt{(3(p + q))^2 - 3(3(p^2 + q^2)}}{3}$$

$$n_{1,}\, n_2 = \frac{3(p + q) \pm \sqrt{9(p + q))^2 - 9(p^2 + q^2)}}{3}$$

$$n_{1,}\, n_2 = \frac{3(p + q) \pm 3\sqrt{(p + q)^2 - (p^2 + q^2)}}{3}$$

$$n_{1,}\, n_2 = \underline{(p + q) \pm \sqrt{(p + q)^2 - (p^2 + q^2)}}$$

$$n_{1,}\, n_2 = (p + q) \pm \sqrt{p^2 + 2pq + q^2 - p^2 - q^2)}$$

$$n_{1,}\, n_2 = (p + q) \pm \sqrt{2pq}$$

Which are the roots for the quadratic not the cubic, so not much gained by this analysis.

An alternative tactic was to compare the coefficients of the quadratic, but this also leads to the same result.

Comparing coefficients; $3n^2 = 3n^2$, $6n(p + q) = 2n(\alpha + \beta + \gamma)$ and $3(p^2 + q^2) = (\alpha\beta + \alpha\gamma + \beta\gamma)$

i.e. $3(p + q) = (\alpha + \beta + \gamma)$ and $3(p^2 + q^2) = (\alpha\beta + \alpha\gamma + \beta\gamma)$

Substituting into the quadratic formula;

$$n_{1,}\, n_2 = \frac{3(p + q) \pm \sqrt{(3(p + q))^2 - 3(3(p^2 + q^2)}}{3}$$

$$n_{1,}\, n_2 = \frac{3(p + q) \pm \sqrt{9(p + q))^2 - 9(p^2 + q^2)}}{3}$$

$$n_{1,}\, n_2 = \frac{3(p + q) \pm 3\sqrt{(p + q)^2 - (p^2 + q^2)}}{3}$$

$$n_{1,}\, n_2 = (p + q) \pm \sqrt{p^2 + 2pq + q^2 - p^2 - q^2)}$$

$$n_{1,}\, n_2 = (p + q) \pm \sqrt{2pq}$$

Which are the roots of the quadratic and doesn't tell us anything more than we already knew and certainly no more about the roots of the cubic.

Appendix 2.4) Different Approach.

The general form of the cubic equation is: $n^3 - n^2(\alpha + \beta + \gamma) + n(\alpha\beta + \alpha\gamma + \beta\gamma) - \alpha\beta\gamma = 0$

Differentiating; $d/dn = 3n^2 - 2n(\alpha + \beta + \gamma) + (\alpha\beta + \alpha\gamma + \beta\gamma) = 0$

Using the quadratic formula to solve this equation gives:

$$n_1, n_2 = \frac{-(-2(\alpha + \beta + \gamma) \pm\sqrt{(-2(\alpha + \beta + \gamma))^2 - 4(3)(\alpha\beta + \alpha\gamma + \beta\gamma)}}{2(3)}$$

$$n_1, n_2 = \frac{2(\alpha + \beta + \gamma) \pm\sqrt{4(\alpha + \beta + \gamma)^2 - 12(\alpha\beta + \alpha\gamma + \beta\gamma)}}{6}$$

$$n_1, n_2 = \frac{2(\alpha + \beta + \gamma) \pm\sqrt{4((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma))}}{6}$$

$$n_1, n_2 = \frac{2(\alpha + \beta + \gamma) \pm 2\sqrt{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma)}}{6}$$

$$n_1, n_2 = \frac{(\alpha + \beta + \gamma) \pm\sqrt{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma)}}{3}$$

We know that the real roots of the quadratic are: $n_1, n_2 = (p + q) \pm\sqrt{2pq}$

Therefore we can equate:

$$(p + q) = (\alpha + \beta + \gamma)/3 \text{ and } \pm\sqrt{2pq} = (\pm\sqrt{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma)})/3$$

In other words: $3(p + q) = (\alpha + \beta + \gamma)$ and $2pq = ((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma))/9$

i.e. : $3(p + q) = (\alpha + \beta + \gamma)$ and $18pq = (\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \alpha\gamma + \beta\gamma)$

But $\{3(p + q)\}^2 = (\alpha + \beta + \gamma)^2$ and $3(p^2 + q^2) = (\alpha\beta + \alpha\gamma + \beta\gamma)$

Therefore $18pq = \{3(p + q)\}^2 - 3(3(p^2 + q^2))$

$$18pq = 9(p + q)^2 - 9(p^2 + q^2)$$

$$18pq = 9(p^2 + 2pq + q^2) - 9(p^2 + q^2)$$

$$18pq = 9p^2 + 18pq + 9q^2 - 9p^2 - 9q^2$$

$$18pq = 18pq$$

which is not a very interesting result and again tells us nothing about the cubic.

Appendix 2.5) Another Approach.
Assume three real roots, $(n - A) = 0$, $(n - B + C) = 0$, $(n - B - C) = 0$

$$\text{Then } (n - A)(n - B + C)(n - B - C) = 0$$

$$(n - A)(n^2 - Bn - Cn - Bn + B^2 + BC + Cn - BC - C^2) = 0$$

$$(n - A)(n^2 - 2Bn + B^2 - C^2) = 0$$

$$(n^3 - 2Bn^2 + nB^2 - nC^2 - An^2 + 2ABn - AB^2 + AC^2) = 0$$

$$(n^3 - n^2 (A + 2B) + n(2AB + B^2 - C^2) - A(B^2 - C^2) = 0$$

Comparing coefficients with the original cubic equation of Eq 3:

$$n^3 - 3n^2(p + q) + 3n(p^2 + q^2) - (p^3 + q^3) = 0 \qquad \text{(Eq 3)}$$

$$(A + 2B) = 3(p + q) \quad \text{which suggests } A = B = (p + q)$$

$$3(p^2 + q^2) = (2AB + B^2 - C^2) \text{ but if } A = B$$

$$\text{then } 3(p^2 + q^2) = (2B^2 + B^2 - C^2) \quad \blacktriangleright \quad 3(p^2 + q^2) = (3B^2 - C^2)$$

$$\text{but } (p^2 + q^2) = ((p + q)^2 - 2pq) \quad \blacktriangleright \quad 3(p^2 + q^2) = 3(p + q)^2 - 6pq$$

And hence $(3B^2 - C^2) = 3(p + q)^2 - 6pq$, which supports $B = (p + q)$ and suggests $C = \pm\sqrt{6pq}$

$$\text{Finally, } A(B^2 - C^2) = (p^3 + q^3), \qquad \text{but } (p + q)^3 = p^3 + q^3 + 3pq(p + q)$$

$$\text{Therefore } (p^3 + q^3) = (p + q)((p + q)^2 - 3pq)$$

$$\text{i.e.} \qquad A(B^2 - C^2) = (p + q)((p + q)^2 - 3pq)$$

which appears to confirm $A = B = (p + q)$, but now suggests $C = \pm\sqrt{3pq}$ which contradicts the early result when $C = \pm\sqrt{6pq}$

This result suggests the original assumption about the nature of the roots was wrong.

Note that changing the assumption for imaginary roots would not make any difference since the only term affected would be C, changing from $- C^2$ to $+ C^2$ in the equation.